

# Well-founded orders in the transfinite Japaridze algebra II

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## Abstract

The logic **GLP** is a polymodal logic that has for each ordinal  $\alpha$  an operator  $[\alpha]$ , whose intended interpretation is a provability predicate in a hierarchy of theories of increasing strength. Its corresponding algebra is called the (transfinite) Japaridze algebra. There are various natural orders in this algebra that are based on comparing consistency strength of its elements. In particular, for each  $\alpha$  we define  $A <_\alpha B \Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A$ .

In this paper we shall consider *worms*, which are formulas of the form  $\langle \alpha_0 \rangle \dots \langle \alpha_n \rangle \top$ , and the partial orders  $<_\alpha$  on their images in the Japaridze algebra. Given a worm  $A$  and an ordinal  $\alpha$ , our goal is to show how one computes the order type that is naturally associated to  $\Omega_\alpha(A) := \{B : B <_\alpha A\}$ .

Our main results show how the sequences  $\langle \Omega_\alpha(A) \rangle_{\alpha \in \text{Ord}}$  can be computed via hyperations and cohyperations, which are forms of transfinite iterations of ordinal functions closely related to Veblen hierarchies.

## 1 Introduction

Below we shall give a precise definition of the Gödel-Löb Polymodal logic **GLP**, a provability logic that has for each ordinal  $\alpha$  a modality  $[\alpha]$ . The idea is that for  $\alpha < \beta$ , the operator  $[\beta]$  represents provability in a theory stronger than that corresponding to  $[\alpha]$ , so that  $[\alpha]\psi \rightarrow [\beta]\psi$  holds. We shall write  $\text{GLP}_\Lambda$  for the part of **GLP** where one only has modalities  $[\alpha]$  for  $\alpha < \Lambda$ .

The logic  $\text{GLP}_\omega$  was first introduced by Japaridze in [5], where  $[n]$  was read as “provable with  $n$  applications of the  $\omega$ -rule”. Later, Ignatiev studied  $\text{GLP}_\omega$  in more detail in [10], showing it arithmetically complete for other readings too. In particular  $\text{GLP}_\omega$  describes the logic where  $[n]$  is read as “provable in elementary arithmetic together with all true  $\Pi_n$ -sentences”. The latter is a  $\Sigma_{n+1}$  statement so that the formula

$$\langle n \rangle \psi \rightarrow [n+1] \langle n \rangle \psi$$

is easily seen to be arithmetically valid. As usual,  $\langle n \rangle \psi$  stands for the  $n$ -consistency of  $\psi$  which is the dual to  $n$ -provability:  $\langle n \rangle \psi := \neg[n]\neg\psi$ .

Lately, interest in the logics  $\text{GLP}_\Lambda$  has revived since Beklemishev applied  $\text{GLP}_\omega$  to give a  $\Pi_1^0$  ordinal analysis of Peano Arithmetic, its sub-theories and some simple extensions (see [2]). This paradigm of  $\Pi_1^0$  ordinal analysis looks very promising as it is more fine-grained than other proof theoretic ordinals ( $\Pi_2^0$ ,  $\Pi_1^1$ ) and can distinguish the proof theoretic strength of, for example, PA and  $\text{PA} + \text{Con}(\text{PA})$  where the others cannot.

However, the theories which allow for a  $\Pi_1^0$  ordinal analysis so far have not been very strong. For theories considerably stronger than PA the machinery needs to be enhanced. A first step in this direction was pursued in [3] where the logics  $\text{GLP}_\Lambda$  were introduced. This paper can be seen as a natural continuation of [3] in that we generalize, analyze and characterize the orders introduced there.

All orderings we shall introduce are important in the general project of applying  $\text{GLP}_\Lambda$  to ordinal analysis: they are closely related to the ordinal representation systems and the meta-mathematical properties of the progressions that arise when transfinitely iterating consistency assertions over some base theory.

## 1.1 The logics $\text{GLP}_\Lambda$

In the definition below the  $\alpha$  and  $\beta$  range over ordinals and the  $\psi$  and  $\chi$  over formulas in the language of  $\text{GLP}_\Lambda$ . The language of  $\text{GLP}_\Lambda$  is that of propositional modal logic that contains for each  $\alpha < \Lambda$  a unary modal operator  $[\alpha]$ .

**Definition 1.1.** *For  $\Lambda$  an ordinal, the logic  $\text{GLP}_\Lambda$  is the propositional normal modal logic that has for each  $\alpha < \Lambda$  a modality  $[\alpha]$  and is axiomatized by the following schemata:*

$$\begin{aligned} &[\alpha](\chi \rightarrow \psi) \rightarrow ([\alpha]\chi \rightarrow [\alpha]\psi), \\ &[\alpha]([\alpha]\chi \rightarrow \chi) \rightarrow [\alpha]\chi, \\ &\langle \alpha \rangle \psi \rightarrow [\beta] \langle \alpha \rangle \psi && \text{for } \alpha < \beta, \\ &[\alpha]\psi \rightarrow [\beta]\psi && \text{for } \alpha \leq \beta. \end{aligned}$$

*The rules of inference are Modus Ponens and necessitation for each modality:  $\frac{\psi}{[\alpha]\psi}$ . By  $\text{GLP}$  we denote the class-size logic that has a modality  $[\alpha]$  for each ordinal  $\alpha$  and all the corresponding axioms and rules.*

It is good to recall that from Löb's axiom  $[\alpha]([\alpha]\chi \rightarrow \chi) \rightarrow [\alpha]\chi$  one can easily derive transitivity, that is,

$$[\alpha]\chi \rightarrow [\alpha][\alpha]\chi,$$

and we shall use this freely in our reasoning.

## 1.2 Worms and the closed fragment of GLP

It turns out that most calculations needed for a  $\Pi_1^0$ -ordinal analysis can be performed in the *closed fragment* of GLP. A closed formula in the language of GLP is simply a formula without propositional variables. In other words, closed formulas are generated by just  $\top$  and the Boolean and modal operators.

The closed fragment of GLP is just the class of closed formulas provable in GLP and is denoted by  $\text{GLP}^0$ . Within this closed fragment and the corresponding algebra, there is a particular class of privileged inhabitants/terms which are called *worms*.

**Definition 1.2** (Worms,  $S$ ,  $S_\alpha$ ). *By  $S$  we denote the set of worms of GLP which is inductively defined as  $\top \in S$  and  $A \in S \Rightarrow \langle \alpha \rangle A \in S$ . Similarly, we inductively define for each ordinal  $\alpha$  the set of worms  $S_\alpha$  where all ordinals are at least  $\alpha$  as  $\top \in S_\alpha$  and  $A \in S_\alpha \wedge \beta \geq \alpha \Rightarrow \langle \beta \rangle A \in S$ .*

Both the closed fragment of GLP and the set of worms have been studied to quite some extent in [3] and [1]. Worms can be conceived as the backbone of  $\text{GLP}^0$ . It is known that each closed formula of GLP is equivalent to a Boolean combination of worms. Moreover, the closed fragment of  $\text{GLP}_\Lambda$  is decidable for decidable  $\Lambda$  and a decision procedure for  $\text{GLP}_\Lambda^0$  proceeds via a reduction to worms. Moreover, in the closed fragment various axioms can be restricted to worms rather than arbitrary closed formulas.

We shall identify a worm  $A$  in the obvious way with  $\iota(A)$ , the string of ordinals in the consistency statements that is involved in  $A$ :  $\iota(\top) = \lambda$  and  $\iota(\langle \alpha \rangle A) = \alpha * \iota(A)$ . In this paper  $\lambda$  will denote the empty string. Worms can thus be perceived as strings over the ordinals and for this reason are also sometimes called *words*. We call them worms here as to refer to the heroic worm-battle, a variant of the Hydra battle (see [4]).

Apart from identifying a worm with its corresponding string of ordinals we shall use any hybrid combination in between at times. For example, we might equally well write  $\omega 0 \omega$ , as  $\langle \omega \rangle 0 \omega$ , or  $\langle \omega \rangle \langle 0 \rangle \langle \omega \rangle \top$ .

The following lemma follows easily from the axioms of GLP and shall be used repeatedly without explicit mention in the remainder of this paper.

**Lemma 1.3.**

1. For a GLP formula  $\phi$  and a worm  $B$ , if  $\beta < \alpha$ , then  
 $\text{GLP} \vdash (\langle \alpha \rangle \phi \wedge \langle \beta \rangle B) \leftrightarrow \langle \alpha \rangle (\phi \wedge \langle \beta \rangle B)$ ;
2. If  $A \in S_{\alpha+1}$ , then  $\text{GLP} \vdash A \wedge \langle \alpha \rangle B \leftrightarrow A \alpha B$ ;
3. If  $A, B \in S_\alpha$  and  $\text{GLP} \vdash A \leftrightarrow B$ , then  
 $\text{GLP} \vdash A \alpha C \leftrightarrow B \alpha C$ .

*Proof.* The  $\rightarrow$  direction of the first item follows from the axiom  $\langle \beta \rangle B \rightarrow [\alpha] \langle \beta \rangle B$ . For the other direction we observe that  $\langle \alpha \rangle \langle \beta \rangle B \rightarrow \langle \beta \rangle B$  in virtue of axiom  $\langle \alpha \rangle \langle \beta \rangle B \rightarrow \langle \beta \rangle \langle \beta \rangle B$  and transitivity of  $[\beta]$ . The other two items follow directly from the first.  $\square$

### 1.3 Plan of the paper

After the introduction, in Section 2 we will revisit some standard notions from ordinal arithmetic that are needed throughout the rest of the paper.

In Section 3 we describe the linear orders  $<_\alpha$  on  $S_\alpha$  defined as  $A <_\alpha B :\Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A$ . The function  $o$  will map a worm to the order type of the set  $\{B \in S \mid B <_0 A\}$ . We resume a calculus for computing  $o$  as presented in [7]. An important ingredient in this calculus is the function  $e^\alpha$  which is defined as the function that enumerates  $o(S_\alpha)$ . The functions  $e^\alpha$  can be seen as a transfinite iterate that we call *hyperation*.

Next, in Section 4 we study the order  $<_\alpha$  on  $S$  in general and not only on  $S_\alpha$ . In this case  $<_\alpha$  no longer linearly orders  $S$  but rather defines a well-founded relation. By  $\Omega_\alpha(A)$  we will denote the supremum of order-types of linear orders that reside in  $\{B \in S \mid B <_\alpha A\}$ . We shall see how the study of  $\Omega_\alpha$ 's can be recursively reduced to the study of  $o_\xi$ 's.

In Section 5 we shall study the sequences  $\langle \Omega_\alpha(A) \rangle_{\alpha \in \text{On}}$  for worms  $A$  and give a full characterization of these sequences. The characterization of these omega sequences is of local nature.

In Section 7 we summarize results from the theory of hyperations and discuss ways to obtain left-inverses of hyperations by means of what we call *cohyperations*.

Finally, in Section 8 we set the cohyperations at work to obtain a global characterization of the omega sequences.

## 1.4 Notation

We reserve lower-case Greek letters  $\alpha, \beta, \gamma, \dots \xi \dots$  for variables ranging over ordinals. Worms will be denoted by upper case latin letters  $A, B, C, \dots$ . The Greek lower-case letters  $\phi, \psi, \chi, \dots$  will denote formulas. However,  $\varphi$  shall be reserved for the Veblen enumeration function and variants thereof. Likewise, we reserve  $\omega$  to denote the first infinite ordinal.

## 2 Ordinal arithmetic

As we shall study well-orders, we need quite some ordinal arithmetic. In this section we shall just state without proof the main properties that we need. For further definitions and detailed proofs, we refer the reader to [11]. Ordinals are canonical representatives for well-orders. The first infinite ordinal is as always denoted by  $\omega$ .

Most operations on natural numbers can be extended to ordinal numbers, like addition, multiplication and exponentiation (see [11]). However, in the realm of ordinal arithmetic things become often more subtle. For example  $1 + \omega = \omega \neq \omega + 1$  and also the other operations differ considerably from ordinary arithmetic.

However, there are various similarities too. In particular we have a form of subtraction available in ordinal arithmetic.

**Lemma 2.1.**

1.  $\forall \zeta < \xi \exists! \eta \zeta + \eta = \xi$   
(We will denote this unique  $\eta$  by  $-\zeta + \xi$ ),
2.  $\forall \eta > 0 \exists \alpha \exists! \beta \eta = \alpha + \omega^\beta$   
(We will denote this unique  $\beta$  by  $\ell \eta$ ),
3.  $\forall \eta > 0 \exists! \alpha, \beta \eta = \omega^\alpha + \beta$  such that  $\beta < \omega^\alpha + \beta$ .

One of the most useful ways to represent ordinals is through their Cantor Normal Forms (CNFs):

**Theorem 2.2** (Cantor Normal Form Theorem).

For each ordinal  $\alpha$  there are unique ordinals  $\alpha_1 \geq \dots \geq \alpha_n$  such that

$$\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}.$$

We call a function  $f$  *increasing* if  $\alpha < \beta$  implies  $f(\alpha) < f(\beta)$ . An ordinal function is called *continuous* if  $\bigcup_{\zeta < \xi} f(\zeta) = f(\xi)$  for all limit ordinals  $\xi$ . Functions which are both increasing and continuous are called *normal*.

It is not hard to see that each normal function has an unbounded set of fixpoints. For example the first fixpoint of the function  $\varphi_0 : x \mapsto \omega^x$  is

$$\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

and is denoted  $\varepsilon_0$ . Clearly for these fixpoints, CNFs give little information as, for example,  $\varepsilon_0 = \omega^{\varepsilon_0}$ . Therefore, we shall need notations and normal forms that are slightly more informative and which are based on functions that enumerate the fixpoints of normal functions: Veblen Normal Forms (VNFs).

In his seminal paper [12], Veblen considered for each normal function  $f$  its derivative  $f'$  that enumerates the fixpoints of  $f$ . If  $f$  is a normal function, then the image of  $f$ —which we shall denote by  $F$ —is a closed (under taking suprema) unbounded set. Likewise the function that enumerates a closed unbounded set is continuous. For  $f$  a normal function, we define  $F'$  to be the image of  $f'$  and we extend this transfinitely by setting

$$\begin{aligned} F_{\alpha+1} &:= (F_\alpha)'; \\ F_\lambda &:= \bigcap_{\alpha < \lambda} F_\alpha \quad \text{for limit } \lambda, \end{aligned}$$

then taking  $f_\lambda$  to be the function that enumerates  $F_\lambda$ .

By taking  $\Phi_0 := \{\omega^\alpha \mid \alpha \in \mathbf{On}\}$  one obtains Veblen's original hierarchy and the  $\varphi_\alpha$  denote the corresponding enumeration functions of the classes  $\Phi_\alpha$ .

Beklemishev noted in [3] that in the setting of GLP it is desirable to have  $1 \notin X_0$ . Thus he considered the progression that started with  $\Phi_0^B := \{\omega^{1+\alpha} \mid \alpha \in \mathbf{On}\}$ . We denote the corresponding enumeration functions by  $\hat{\varphi}_\alpha$ .

In [9] and in this paper the authors realized that, moreover it is desirable to have 0 in the initial set, whence we departed from

$$E_0 = \{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \mathbf{On}\}.$$

We shall denote the corresponding enumeration functions by  $e_\alpha$ . In general, if  $f$  is some normal function, we shall denote by  $f_\alpha$  the Veblen progression based on  $f_0 = f$ . Note that, if  $\alpha < \beta$ , we have that  $f_\beta(\gamma)$  is always a fixpoint of  $f_\alpha$ , i.e.,  $f_\beta = f_\alpha \circ f_\beta$ .

One readily observes that

$$\begin{aligned} e_\alpha(0) &= 0 && \text{for all } \alpha; \\ e_0(1 + \beta) &= \varphi_0(1 + \beta) = \hat{\varphi}_0(\beta) && \text{for all } \beta; \\ e_{1+\alpha}(1 + \beta) &= \varphi_{1+\alpha}(\beta) = \hat{\varphi}_{1+\alpha}(\beta) && \text{for all } \alpha, \beta. \end{aligned}$$

Many times, we can write an ordinal  $\omega^\alpha$  in more than one way as  $\varphi_\xi(\eta)$ . However, if we require that  $\eta < \varphi_\xi(\eta)$ , then both  $\xi$  and  $\eta$  are uniquely determined. In other words

$$\forall \alpha \exists! \eta, \xi [\omega^\alpha = \varphi_\xi(\eta) \wedge \eta < \varphi_\xi(\eta)].$$

Combining this fact with the CNF Theorem one obtains *Veblen Normal Forms* for ordinals.

**Theorem 2.3** (Veblen Normal Form Theorem). *For all  $\alpha$  there exist unique  $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$  ( $n \geq 0$ ) such that*

1.  $\alpha = \varphi_{\alpha_1}(\beta_1) + \dots + \varphi_{\alpha_n}(\beta_n)$ ,
2.  $\varphi_{\alpha_i}(\beta_i) \geq \varphi_{\alpha_{i+1}}(\beta_{i+1})$  for  $i < n$ ,
3.  $\beta_i < \varphi_{\alpha_i}(\beta_i)$  for  $i \leq n$ .

Note that  $\alpha_i \geq \alpha_{i+1}$  does not in general hold in the VNF of  $\alpha$ . For example,

$$\omega^{\varepsilon_0+1} + \varepsilon_0 = \varphi_0(\varepsilon_0 + 1) + \varphi_1(0) = \varphi_0(\varphi_{\varphi_0(0)}(0) + \varphi_0(0)) + \varphi_{\varphi_0(0)}(0).$$

### 3 Linear orders on the Japaridze algebra

In this section we shall introduce linear orders on worms, an important theme in our paper.

#### 3.1 The orderings $<_\alpha$

It is known that the class of worms is linearly ordered by consistency strength. That is, two worms are either equivalent or one of the two implies the consistency (0-consistency that is) of the other.

**Definition 3.1** ( $<, <_\alpha, o, o_\alpha$ ). *We define a relation  $<_\alpha$  on  $S_\alpha \times S_\alpha$  by*

$$A <_\alpha B :\Leftrightarrow \text{GLP} \vdash B \rightarrow \langle \alpha \rangle A \quad (\text{with } A, B \in S_\alpha).$$

For  $A \in S_\alpha$  we denote by  $o_\alpha(A)$  the order type of  $\{B \in S_\alpha \mid B <_\alpha A\}$ . More precisely, for  $A \in S_\alpha$  we define inductively

$$o_\alpha(A) = \sup \{o_\alpha(B) + 1 : B \in S_\alpha \text{ \& } B <_\alpha A\},$$

where  $\sup \emptyset = 0$ .

When  $X$  is a set or class we shall denote by  $o_\alpha(X)$  the image of  $X$  under  $o_\alpha$ .

Instead of  $<_0$  and  $o_0$  we shall write  $<$  and  $o$ , respectively.

### 3.2 Japaridze algebras

The relations  $<_\alpha$  do not give proper linear orders on  $S_\alpha$ , given that different worms may be equivalent and hence undistinguishable in the ordering. We remedy this by passing to the Lindenbaum algebra of GLP – that is, the quotient of the language of GLP modulo provable equivalence.

This algebra is a *Japaridze algebra*, as described below:

**Definition 3.2** (Japaridze algebra). *A Japaridze algebra is a structure*

$$\mathcal{J} = \langle D, \{[\alpha]\}_{\alpha < \Lambda}, \wedge, \neg, 0, 1 \rangle$$

*such that*

1.  $\langle D, \wedge, \neg, 0, 1 \rangle$  is a Boolean algebra,
2.  $[\alpha]1 = 1$  for all  $\alpha < \Lambda$ ,
3.  $[\alpha](x \rightarrow y) \leq [\alpha]x \rightarrow [\alpha]y$  for all  $\alpha < \Lambda$ ,  $x, y \in D$ ,
4.  $[\alpha]([\alpha]x \rightarrow x) \leq [\alpha]x$  for all  $\alpha < \Lambda$ ,  $x \in D$ ,
5.  $[\alpha]x \leq [\beta]x$  for all  $\alpha \leq \beta < \Lambda$ ,  $x \in D$  and,
6.  $\langle \alpha \rangle x \leq [\beta] \langle \alpha \rangle x$  for all  $\alpha < \beta < \Lambda$ ,  $x \in D$ ,

where  $\langle \alpha \rangle, \rightarrow$  are defined in the usual way.

It is in these algebras that the partial orders  $<_\alpha$  we have described naturally reside. However, rather than work with abstract elements of the Lindenbaum algebra, it will be convenient to choose suitable representatives of each equivalence class. These representatives for terms without variables will be given by *Beklemishev Normal Forms* (BNFs), as described below.

### 3.3 A well-order on Beklemshev Normal Forms

BNFs are a subclass of  $S$  on which  $<_0$  does define a linear order as was shown in [1, 3]. In those papers it was also shown that each worm is equivalent to a unique worm in BNF and that this BNF can be found effectively for recursive well-orders. Moreover, if  $A \in S_\alpha$ , then its equivalent in BNF is also in  $S_\alpha$ .

In this section we shall provide a calculus to compute  $o_\alpha$ . Note that it is not at all obvious that  $o_\alpha$  is defined everywhere, but this turns out to be the case.

**Definition 3.3** (Beklemishev Normal Form). *A worm  $A \in S$  is in BNF (Beklemishev Normal Form) iff*

1.  $A = \lambda$  or,
2.  $A$  is of the form  $A_k \alpha \dots \alpha A_1$  with  $\alpha = \min(A)$ ,  $k \geq 1$  and  $A_i \in S_{\alpha+1}$  such that each  $A_i$  is in BNF and moreover  $A_{i+1} \leq_{\alpha+1} A_i$  for each  $i < k$ .

We shall write  $\mathcal{B}$  for BNF and  $\mathcal{B}_\alpha$  for  $\text{BNF} \cap S_\alpha$ .

**Lemma 3.4.** *Each worm of the form  $\alpha^n$ , i.e.,  $\overbrace{\langle \alpha \rangle \dots \langle \alpha \rangle}^{n \text{ times}} \top$ , is in BNF.*

*Proof.* This is immediate if we conceive  $\alpha^n$  as  $\lambda \alpha \lambda \dots \lambda \alpha \lambda$ .  $\square$

As announced before, the BNFs form a class of natural representatives for formulas without variables with respect to  $o$ :

**Lemma 3.5.** *The map  $o : (\mathcal{B}, <_0) \rightarrow (\text{Ord}, <)$  defines an isomorphism.*

The definition of BNFs –Definition 3.3– reveals a strong similarity between BNFs and CNFs and throughout the paper we shall see more analogies.

### 3.4 A calculus for $o$

In this subsection we state a calculus for computing  $o$  and  $o_\alpha$ . Proofs and details of the calculus presented here can be found in [7]. We first need a syntactical operation that promotes or demotes worms in terms of consistency strength.

**Definition 3.6** ( $\alpha \uparrow$  and  $\alpha \downarrow$ ). *Let  $A$  be a worm and  $\alpha$  an ordinal. By  $\alpha \uparrow A$  we denote the worm that is obtained by simultaneously substituting each  $\beta$  that occurs in  $A$  by  $\alpha + \beta$ .*

*Likewise, if  $A \in S_\alpha$  we denote by  $\alpha \downarrow A$  the worm that is obtained by replacing simultaneously each  $\beta$  in  $A$  by  $-\alpha + \beta$ .*

Note that by Lemma 2.1, the operation  $\alpha \downarrow$  is well-defined on  $S_\alpha$ . The next lemma enumerates some noteworthy properties of these promoting and demoting operations. Proofs for the non-trivial items can be found in [7].

**Lemma 3.7.** *For  $\alpha, \beta, \gamma$  ordinals and worms  $A, B$  we have:*

1.  $\alpha \uparrow \beta < \alpha \uparrow \gamma \Leftrightarrow \beta < \gamma$ ,
2.  $\alpha \uparrow \beta \geq \beta$ ,
3.  $\alpha \uparrow (\beta \uparrow A) = (\alpha + \beta) \uparrow A$ ,
4.  $\alpha \downarrow (\beta \uparrow A) = (-\alpha + \beta) \uparrow A$ , provided  $\alpha \leq \beta$ ,
5.  $\alpha \downarrow (\beta \downarrow A) = (\beta + \alpha) \downarrow A$ , provided  $A \in S_{\beta+\alpha}$ ,
6.  $\alpha \uparrow ((\beta + \alpha) \downarrow A) = \beta \downarrow A$  for  $A \in S_\alpha$ ,
7.  $A <_\alpha B \Leftrightarrow A < B$  for  $A, B \in S_\alpha$ ,



$$8. A <_{\xi} B \Leftrightarrow \alpha \uparrow A <_{\alpha+\xi} \alpha \uparrow B.$$

Moreover, in [7] it is proven that  $\alpha \uparrow$  is a well-behaved map with nice properties. In Lemma 3.8 below we see that  $\alpha \uparrow$  can be viewed as continuous on  $S$  in that taking suprema commutes with  $\alpha \uparrow$ .

**Lemma 3.8.** *For any set  $\{A_i\}_{i \in I}$  of worms we have that*

$$\alpha \uparrow \sup_{i \in I} A_i = \sup_{i \in I} \alpha \uparrow A_i.$$

Moreover,  $\alpha \uparrow$  can also be viewed as an isomorphism:

**Lemma 3.9.** *The map  $\alpha \uparrow$  is an isomorphism between  $(S, <)$  and  $(S_{\alpha}, <_{\alpha})$ .*

The following function on ordinals turns out to be closely related to  $\alpha \uparrow$ :

**Definition 3.10** ( $e^{\alpha}$ ). *We define  $e^{\alpha}$  to be the function that enumerates  $o(S_{\alpha})$ .*

We call these function  $e^{\alpha}$  *hyperexponentials* as they can be perceived as transfinitely iterating  $e_0$  which is a particular form of exponentiation. The function  $o$  is defined on all worms. Par abus de langage we will denote the restriction of  $o$  to  $\mathcal{B}$  also by  $o$ . In particular, whenever we write  $o^{-1}$  it is tacitly understood that we refer to the restriction of  $o$  to  $\mathcal{B}$ .

**Lemma 3.11.**  *$o(S_{\alpha})$  is enumerated by  $o \circ \alpha \uparrow \circ o^{-1}$ , that is,*

$$e^{\alpha} = o \circ \alpha \uparrow \circ o^{-1}.$$

An explicit recursive scheme to compute the values of  $e^{\alpha}\beta$  is given in the following theorem:

**Theorem 3.12.** *For ordinals  $\alpha$  and  $\beta$ , the values  $e^{\alpha}(\beta)$  are determined by the following recursion.*

1.  $e^{\alpha}0 = 0$  for all  $\alpha \in \text{Ord}$ ;
2.  $e^1 = e$  where  $e$  enumerates the set  $\{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \text{Ord}\}$ ;
3.  $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$ ;
4.  $e^{\alpha}(\lambda) = \cup_{\beta < \lambda} e^{\alpha}(\beta)$  for limit ordinals  $\lambda$ ;
5.  $e^{\lambda}(\beta + 1) = \cup_{\lambda' < \lambda} e^{\lambda'}(e^{\lambda}(\beta) + 1)$  for  $\lambda$  an additively indecomposable limit ordinal.

Based on these hyperexponential functions  $e^{\alpha}$  we can formulate an elegant calculus to compute the values of  $o_{\alpha}(A)$ :

**Theorem 3.13.**

1.  $o(0^n) = n$ ;

2. If  $A = A_n 0 \dots A_1 \in \mathcal{B}$  and  $A_1 \in \mathcal{B}_1$  is not empty, then  
 $o(A) = \omega^{o(1 \downarrow A_1)} + \dots + \omega^{o(1 \downarrow A_n)}$ , where  
for  $n = 1$  we denote by  $A_n 0 \dots A_1$  simply  $A_1$ ;
3.  $o(\xi \uparrow A) = e^\xi o(A)$ ,
4.  $o_\xi(A) = o(\xi \downarrow A)$  for  $A \in S_\xi$ .

Note that the last item of this theorem is not needed to compute  $o$ . It merely tells us how to reduce  $o_\alpha$  to  $o$ . The  $e^\alpha$  functions can be related to the more familiar Veblen progressions.

**Lemma 3.14.**  $e^{\omega^\alpha} = e_\alpha$ .

Moreover, we note that Lemma 3.14 together with Theorem 3.12.3 yields a reduction of computing  $e^\alpha$  to the better known Veblen-like functions  $e_\alpha$ . For if  $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ , then

$$e^\alpha = e_{\alpha_1} \circ \dots \circ e_{\alpha_n}.$$

## 4 Well-founded orders on worms

In this section we consider the ordering  $<_\alpha$  on the full  $S \times S$  rather than on  $S_\alpha \times S_\alpha$ . We shall see that the resulting order is still well-founded but no longer total.

### 4.1 Well-founded orders

In Section 3 we presented the well-orders  $<_\alpha$  on  $S_\alpha$ . We can also consider the ordering  $<_\alpha$  on the full class  $S$ . As we shall see,  $<_\alpha$  is no longer linear on  $S$ . However, it is still well-founded. Anticipating this, we can define  $\Omega_\alpha(A)$ , the generalized  $<_\alpha$  order-type of a worm  $A$ .

**Definition 4.1.** Given an ordinal  $\xi$  and a worm  $A$ , we define a new ordinal  $\Omega_\xi(A)$  inductively on  $<_\xi$  by

$$\Omega_\xi(A) = \sup_{B <_\xi A} (\Omega_\xi(B) + 1).$$

With this, we can assign to each worm  $A$  a sequence of order-types.

We will use the notation  $\vec{\Omega}(A)$  for the sequence  $(\Omega_\xi(A))_{\xi \in \text{On}}$ ; that is,

$$\vec{\Omega}(A) := (\Omega_0(A), \Omega_1(A), \dots, \Omega_\omega(A), \Omega_{\omega+1}(A) \dots).$$

We shall refer to these sequences as *Omega-sequences*.

## 4.2 Omega-sequences and modal semantics

Each worm  $A$  is known to be consistent with GLP, hence should be satisfied in an exact model for its closed fragment, if it exists; that is, a model on which only the theorems of  $\text{GLP}^0$  are valid.

Suppose  $\mathcal{M}$  were such a model. Each modality  $\langle \xi \rangle$  will be represented in  $\mathcal{M}$  by some relation  $\prec_\xi$  in that

$$\mathcal{M}, w \Vdash \langle \xi \rangle \phi \Leftrightarrow \exists w' (w' \prec_\xi w \wedge \mathcal{M}, w' \Vdash \phi).$$

As  $[\xi]$  satisfies Löb's axiom, we know that each  $\prec_\xi$  is transitive and well-founded. Consequently, we can assign to each world  $w$  a sequence of ordinals

$$\vec{w} := (w_0, w_1, \dots, w_\omega, w_{\omega+1} \dots),$$

where  $w_\zeta$  corresponds to the supremum of order-types of  $\prec_\zeta$ -chains below  $w$ . If  $\mathcal{M}, w \Vdash A$ , then necessarily  $w_\xi \geq \Omega_\xi(A)$  for each  $\xi$ . A systematic study of  $\vec{\Omega}(A)$  will thus also reveal information about models for  $\text{GLP}^0$ .

No such models were known, but in [6] the authors define a universal class-size model for  $\text{GLP}^0$ . The worlds in that model closely reflect the  $\Omega_\xi(A)$  sequences as defined here. In particular, it turns out that the necessary condition that if  $\mathcal{M}, w \Vdash A$ , then  $w_\xi \geq \Omega_\xi(A)$  for each  $\xi$  is actually also sufficient.

In Section 5 we shall characterize the sequences  $\Omega_\xi(A)$  for given  $\xi$  and  $A$ . In the next subsection we shall see how questions about  $\Omega_\xi$  can be recursively reduced to questions about  $o_\zeta$ .

## 4.3 Reducing $\Omega_\xi$ to $o_\zeta$

In Lemma 4.4 below we shall see how questions about  $\Omega_\xi$  can be recursively reduced to questions about  $o_\zeta$ . For this reduction we need the syntactical definitions of *head* and *remainder*.

**Definition 4.2.** *Let  $A$  be a worm. By  $h_\xi(A)$  we denote the  $\xi$ -head of  $A$ . Recursively:  $h_\xi(\lambda) = \lambda$ ,  $h_\xi(\zeta * A) = \zeta * h_\xi(A)$  if  $\zeta \geq \xi$  and  $h_\xi(\zeta * A) = \lambda$  if  $\zeta < \xi$ .*

*Likewise, by  $r_\xi(A)$  we denote the  $\xi$ -remainder of  $A$ :  $r_\xi(\lambda) = \lambda$ ,  $r_\xi(\zeta * A) = r_\xi(A)$  if  $\zeta \geq \xi$  and  $r_\xi(\zeta * A) = \zeta * A$  if  $\zeta < \xi$ .*

In words,  $h_\xi(A)$  corresponds to the largest initial part (reading from left to right) of  $A$  such that all symbols in  $h_\xi(A)$  are at least  $\xi$  and  $r_\xi(A)$  is that part of  $A$  that remains when removing its  $\xi$ -head. We thus have  $A = h_\xi(A) * r_\xi(A)$  for all  $\xi$  and  $A$ .

Observe that

$$\text{GLP} \vdash h_\xi(A) * r_\xi(A) \leftrightarrow h_\xi(A) \wedge r_\xi(A), \quad (1)$$

as the first symbol of  $r_\xi(A)$  is less than  $\xi$  and  $h_\xi(A) \in S_\xi$  (see Lemma 1.3). Moreover, for each  $\xi$  and each  $A$  we have that  $h_\xi(A)$  is in normal form whenever  $A$  is:

**Lemma 4.3.** *If  $A \in \text{BNF}$ , then also  $h_\zeta(A) \in \text{BNF}$  and  $r_\zeta(A) \in \text{BNF}$ .*

*Proof.* We prove here the  $h_\zeta(A)$  case. For  $A = \lambda$  this is clear. Thus, let the symbols in  $A$  be enumerated in increasing order by  $\xi_0, \dots, \xi_n$ . By an easy induction on  $n$  we see that each  $h_{\xi_i}(A) \in \text{BNF}$ . If  $\xi_n > \zeta \notin A$ , then  $h_\zeta(A) = h_{\min\{\xi_i \mid \xi_i > \zeta\}}(A)$ . If  $\zeta > \xi_n$ , then  $h_\zeta(A) = \lambda$  which is in  $\text{BNF}$ .  $\square$

**Lemma 4.4.** *Let  $A$  and  $B$  be worms. We have that*

$$(A \rightarrow \langle \xi \rangle B) \Leftrightarrow [(h_\xi(A) \rightarrow \langle \xi \rangle h_\xi(B)) \wedge (A \rightarrow r_\xi(B))].$$

*Proof.* “ $\Rightarrow$ ” By 1,  $B \leftrightarrow h_\xi(B) \wedge r_\xi(B)$  whence  $A \rightarrow r_\xi(B)$  as

$$\begin{aligned} A &\rightarrow \langle \xi \rangle B \\ &\rightarrow \langle \xi \rangle (h_\xi(B) \wedge r_\xi(B)) \quad \text{by Lemma 1.3.2} \\ &\rightarrow r_\xi(B) \wedge \langle \xi \rangle h_\xi(B) \\ &\rightarrow r_\xi(B). \end{aligned}$$

Likewise  $A \leftrightarrow h_\xi(A) \wedge r_\xi(A)$ . As  $h_\xi(A), h_\xi(B) \in S_\xi$  we know that either

- $h_\xi(A) = h_\xi(B)$ ,
- $h_\xi(B) \rightarrow \langle \xi \rangle h_\xi(A)$  or,
- $h_\xi(A) \rightarrow \langle \xi \rangle h_\xi(B)$ .

By assumption  $A \rightarrow \langle \xi \rangle B$  whence  $A \rightarrow \langle \xi \rangle h_\xi(B) \wedge r_\xi(B)$ .

Suppose now  $h_\xi(A) = h_\xi(B)$ . Then,

$$h_\xi(A) \wedge r_\xi(A) \rightarrow \langle \xi \rangle h_\xi(A) \wedge r_\xi(A)$$

whence also

$$h_\xi(A) \wedge r_\xi(A) \rightarrow \langle \xi \rangle (h_\xi(A) \wedge r_\xi(A)).$$

The latter is equivalent to  $A \rightarrow \langle \xi \rangle A$  which contradicts the irreflexivity of  $<_\xi$ .

By a similar argument, the assumption that  $h_\xi(B) \rightarrow \langle \xi \rangle h_\xi(A)$  contradicts the irreflexivity of  $<_\xi$  and we conclude that  $h_\xi(A) \rightarrow \langle \xi \rangle h_\xi(B)$ .

“ $\Leftarrow$ ” This is the easier direction.

$$\begin{aligned} A &\leftrightarrow h_\xi(A) \wedge r_\xi(A) \\ &\rightarrow \langle \xi \rangle h_\xi(B) \wedge r_\xi(B) \\ &\rightarrow \langle \xi \rangle (h_\xi(B) \wedge r_\xi(B)) \\ &\rightarrow \langle \xi \rangle B. \end{aligned}$$

$\square$

Note that this lemma recursively reduces the general  $<_\xi$  question between worms, to the  $<_\xi$  questions between worms in  $S_\xi$ . Note also that  $<_\xi$  is not tree-like; for example, we see that both  $011 <_1 10111 <_1 1111$  and  $011 <_1 11011 <_1 1111$  while  $10111$  and  $11011$  are  $<_1$  incomparable.

In general  $<_\alpha$  does not define a well-quasi-ordering on  $S$ . For example, all elements  $\{\langle \beta \rangle \top \mid \beta < \alpha\}$  are mutually  $<_\alpha$  incomparable. A natural questions to study for the  $<_\alpha$  orderings on  $S \times S$  concerns the  $<_0$  length of anti-chains. So, given a worm  $A$ , we can consider sets  $X_i = \{B \mid B <_\alpha A\}$  so that all elements in  $X_i$  are mutually  $<_\alpha$ -incomparable. The question arises, what is  $\sup_i \text{ot}(X_i, <_0)$ ?

## 5 Omega sequences

In this section we give a full characterization of the sequences  $\vec{\Omega}(A)$ ; that is, we shall determine for given  $A$  each of the values  $\Omega_\xi(A)$  and classify at what coordinates  $\xi$  the  $\vec{\Omega}(A)$  sequence changes value.

### 5.1 Basic properties of omega sequences

Clearly,  $\vec{\Omega}(A)$  defines a weakly decreasing sequence of ordinals.

**Lemma 5.1.** *For  $\xi < \zeta$  we have that  $\Omega_\xi(A) \geq \Omega_\zeta(A)$ .*

*Proof.* In general we have for  $\xi < \zeta$  that  $A \rightarrow \langle \zeta \rangle B$  implies  $A \rightarrow \langle \xi \rangle B$ . Thus, any  $<_\zeta$  sequence is automatically also a  $<_\xi$  sequence.  $\square$

In particular, since the omega sequences are weakly decreasing on the ordinals, we have that  $\{\Omega_\xi(A) \mid \xi \in \text{Ord}\}$  is a finite set for any worm  $A$ .

**Lemma 5.2.**  $\Omega_\xi(A) = o_\xi h_\xi(A)$

*Proof.* Suppose  $A_0 <_\xi A_1 <_\xi \dots <_\xi A$ , then

$$h_\xi(A_0) <_\xi h_\xi(A_1) <_\xi \dots <_\xi h_\xi(A)$$

by Lemma 4.4 whence  $\Omega_\xi(A) \leq o_\xi h_\xi(A)$ .

On the other hand, if  $B <_\xi h_\xi(A)$ , then  $h_\xi(A) \rightarrow \langle \xi \rangle B$ . But as  $A \leftrightarrow h_\xi(A) \wedge r_\xi(A)$  we also have  $A \rightarrow \langle \xi \rangle B$ . Consequently  $o_\xi h_\xi(A) \leq \Omega_\xi(A)$ .  $\square$

**Corollary 5.3.** *For each worm  $A$ , there is a maximal  $\xi$  so that  $\Omega_\xi(A) \neq 0$ . In particular we have  $\xi = \text{First}(A)$ , where  $\text{First}(A)$  is the left-most element of  $A$ .*

*Proof.* For  $A \in S$ , we denote by  $\text{First}(A)$  the first element of  $A$ , that is,  $\text{First}(\lambda) = \lambda$ , and  $\text{First}(\xi * B) = \xi$ . Clearly,  $h_{\text{First}(A)}(A) \neq \lambda$  whence by Lemma 5.2,

$$\Omega_{\text{First}(A)}(A) \neq 0.$$

On the other hand, for  $\xi > \text{First}(A)$ , clearly  $h_\xi(A) = \lambda$  whence  $\Omega_\xi(A) = 0$ .  $\square$

It is good to have reduced  $\Omega_\xi(A)$  to  $o_\xi(A)$  as in Section 3 we provided a full calculus for it (Lemma 3.13).

Lemma 5.1 and Corollary 5.3 are first simple observations on  $\vec{\Omega}(A)$  sequences. In the remainder of this section we shall provide a full characterization of them.

### 5.2 Successor coordinates

First let us compute  $\Omega_{\xi+1}(A)$  in terms of  $\Omega_\xi(A)$ . Recall that  $\ell\alpha$  denotes the unique  $\beta$  such that  $\alpha = \alpha' + \omega^\beta$  for  $\alpha > 0$ . For convenience we define  $\ell 0 = 0$ . The following lemma will be useful:

**Lemma 5.4.** *Given an ordinal  $\xi$  and a worm  $A$ ,*

$$o_{\xi+1}h_{\xi+1}(A) = \ell o_{\xi}h_{\xi}(A).$$

*Proof.* We write  $h_{\xi}(A)$  as  $A_0\xi \dots \xi A_n$ . Clearly,  $h_{\xi+1}(A) = A_0$ . We shall now see that  $\ell o_{\xi}h_{\xi}(A) = o_{\xi+1}(A_0)$ .

To this end, we observe that

$$\begin{aligned} o_{\xi}h_{\xi}(A) &= o_{\xi}(A_0\xi \dots \xi A_n) \\ &= o\left((\xi \downarrow A_0)0 \dots 0(\xi \downarrow A_n)\right) \\ &= \omega^{o_1(\xi \downarrow A_n)} + \dots + \omega^{o_1(\xi \downarrow A_0)} \\ &= \omega^{o_{\xi+1}(A_n)} + \dots + \omega^{o_{\xi+1}(A_0)} \end{aligned}$$

Consequently  $\ell o_{\xi}h_{\xi}(A) = o_{\xi+1}(A_0)$ , as desired.  $\square$

Now we are ready to describe the relation between successor coordinates of the  $\vec{\Omega}(A)$  sequence.

**Theorem 5.5.**  $\Omega_{\xi+1}(A) = \ell\Omega_{\xi}(A)$

*Proof.*

$$\begin{aligned} \Omega_{\xi+1}(A) &= o_{\xi+1}h_{\xi+1}(A) \quad \text{by Lemma 5.4} \\ &= \ell o_{\xi}h_{\xi}(A) \\ &= \ell\Omega_{\xi}(A) \quad \text{by Lemma 4.4.} \end{aligned}$$

$\square$

Theorem 5.5 tells us what the relation between successor coordinates of  $\vec{\Omega}(A)$  is. We may also infer from it when successor coordinates are different; if  $\Omega_{\xi}(A)$  is a fixed point of  $\zeta \mapsto \omega^{\zeta}$  then  $\Omega_{\xi}(A) = \Omega_{\xi+1}(A)$ .

### 5.3 Equal coordinates

Theorem 5.7 below gives us a characterization of when different coordinates attain different or equal values. Before we can state and prove this theorem we first need some notation and background reasoning on CNFs.

For  $\alpha \in \mathbf{On}$  we define  $N_{\alpha}$  and the syntactic operation  $\mathbf{CNF}(\alpha) := \sum_{i=1}^{N_{\alpha}} \omega^{\xi_i}$  to be the unique CNF expression of  $\alpha$ . Next, we define for an ordinal  $\alpha$  the set of its *Cantor Normal Form Approximations* as the set of partial sums of  $\mathbf{CNF}(\alpha)$ , that is, if

$$\mathbf{CNF}(\alpha) = \sum_{i=1}^{N_{\alpha}} \omega^{\xi_i},$$

then

$$\mathbf{CNA}(\alpha) := \left\{ \sum_{i=1}^k \omega^{\xi_i} : 0 \leq k \leq N_{\alpha} \right\}.$$

We also define the *Cantor Normal Form Projection* of some ordinal  $\zeta$  on another ordinal  $\xi$  as follows:

$$\text{CNP}(\zeta, \xi) := \max\{\xi' \in \text{CNA}(\xi) \mid \xi' \leq \zeta\}.$$

Note that  $\text{CNP}(\zeta, \xi)$  is defined for all  $\zeta, \xi \in \text{On}$ .

For  $\alpha, \beta, \gamma \in \text{On}$  we define

$$\alpha \sim_\gamma \beta \iff \text{CNA}(\alpha, \gamma) = \text{CNA}(\beta, \gamma).$$

In words,  $\alpha \sim_\gamma \beta$  whenever there is no partial sum of the CNF of  $\gamma$  that falls in between  $\alpha$  and  $\beta$  (also the case that both  $\alpha$  and  $\beta$  are non-equal partial sums is excluded).

The just-defined notions of  $\text{CNA}(\xi)$ ,  $\text{CNP}(\zeta, \xi)$  and  $\alpha \sim_\gamma \beta$  are needed to characterize the  $\xi \downarrow \zeta$  operation.

**Lemma 5.6.** *Let  $\zeta, \xi$  and  $\eta$  be ordinals.*

$$1. \forall \zeta \leq \xi \quad \zeta \downarrow \xi = \text{CNP}(\zeta, \xi) \downarrow \xi;$$

$$2. \forall \zeta \leq \xi \exists! \eta \in \text{CNA}(\xi) \quad \zeta \downarrow \xi = \eta \downarrow \xi;$$

$$3. \text{ For } \xi, \zeta \leq \eta, \text{ we have } \xi \downarrow \eta = \zeta \downarrow \eta \iff \xi \sim_\eta \zeta.$$

*Proof.* 1. We consider  $\zeta \leq \xi$ . Now let  $\eta = \max\{\eta' \in \text{CNA}(\xi) \mid \eta' \leq \zeta\} = \text{CNP}(\zeta, \xi)$ . The claim is that  $\zeta \downarrow \xi = \eta \downarrow \xi$ . Let

$$\text{CNF}(\xi) = \sum_{i=1}^{N_\xi} \omega^{\xi_i}.$$

As  $\eta = \sum_{i=1}^k \omega^{\xi_i}$  for some  $k \leq N_\xi$ , we see that

$$\eta \downarrow \xi = \sum_{i=k+1}^{N_\xi} \omega^{\xi_i}$$

for  $k < N_\xi$  and  $\eta \downarrow \xi = 0$  for  $k = N_\xi$ . We now claim that  $\zeta + (\eta \downarrow \xi) = \xi$  so that  $\zeta \downarrow \xi = \eta \downarrow \xi$  follows from the fact that

$$\forall \zeta < \xi \exists! \delta \quad \zeta + \delta = \xi.$$

We may assume  $\zeta > \eta$  otherwise  $\zeta + (\eta \downarrow \xi) = \xi$  is trivial.

Thus,

$$\eta = \sum_{i=1}^k \omega^{\xi_i} < \zeta \leq \sum_{i=1}^{k+1} \omega^{\xi_i}.$$

As by the definition of  $\eta$  we see that  $\zeta \leq \sum_{i=1}^{k+1} \omega^{\xi_i}$  cannot be an equality whence

$$\eta = \sum_{i=1}^k \omega^{\xi_i} < \zeta < \sum_{i=1}^{k+1} \omega^{\xi_i}.$$

Thus,  $\eta \in \text{CNA}(\zeta)$  and  $\zeta + \sum_{i=k+1}^{N_\xi} \omega^{\xi_i} = \xi$ , whence

$$\sum_{i=k+1}^{N_\xi} \omega^{\xi_i} = \zeta \downarrow \xi = \sum_{i=1}^k \omega^{\xi_i} = \eta \downarrow \xi.$$

2. Follows from part 1 once we realize that for different  $\eta$  and  $\eta'$  both in  $\text{CNA}(\xi)$  we have  $\eta \downarrow \xi \neq \eta' \downarrow \xi$ .

3. From the proof of part 1 we see that

$$\xi \downarrow \eta = \zeta \downarrow \eta \Leftrightarrow \max\{\eta' \in \text{CNA}(\eta) \mid \eta' \leq \xi\} = \max\{\eta' \in \text{CNA}(\eta) \mid \eta' \leq \zeta\}$$

where the latter is precisely the definition of  $\xi \sim_\eta \zeta$ .  $\square$

Once we have this lemma to characterize the  $\xi \downarrow \zeta$  operation, we are armed to prove a characterization for when two coordinates in  $\bar{\Omega}(A)$  are equal.

**Theorem 5.7.** *The following five conditions are equivalent.*

1.  $\Omega_\xi(A) = \Omega_\zeta(A)$
2.  $o_\xi h_\xi(A) = o_\zeta h_\zeta(A)$
3.  $\xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A)$
4.  $h_\xi(A) = h_\zeta(A)$  and  $\xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A)$
5.  $h_\xi(A) = h_\zeta(A)$  and  $\forall \eta \in h_\xi(A), \xi \sim_\eta \zeta$

*Proof.* (1.)  $\Leftrightarrow$  (2.) is just Lemma 5.2.

(2.)  $\Leftrightarrow$  (3.): Observe that  $o_\xi(h_\xi(A)) = o(\xi \downarrow h_\xi(A))$  and  $o_\zeta(h_\zeta(A)) = o(\zeta \downarrow h_\zeta(A))$ . As  $o$  defines an isomorphism between  $S$  and  $\text{On}$ , we obtain

$$o_\xi h_\xi(A) = o_\zeta h_\zeta(A) \Leftrightarrow \xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A).$$

(3.)  $\Leftrightarrow$  (4.): Suppose  $\xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A)$ . Then, it follows that the two have equal length; further, they have length equal to that of  $h_\xi(A), h_\zeta(A)$ , respectively. But two initial segments of  $A$  of equal length must be equal, that is,  $h_\xi(A) = h_\zeta(A)$ .

(4.)  $\Leftrightarrow$  (5.):

$$\begin{array}{lll} h_\xi(A) = h_\zeta(A) & \& \xi \downarrow h_\xi(A) = \zeta \downarrow h_\zeta(A) & \Leftrightarrow \\ h_\xi(A) = h_\zeta(A) & \& \forall \eta \in h_\xi(A) \xi \downarrow \eta = \zeta \downarrow \eta & \Leftrightarrow \text{ by Lemma 5.6.3} \\ h_\xi(A) = h_\zeta(A) & \& \forall \eta \in h_\xi(A) \xi \sim_\eta \zeta & \end{array}$$

$\square$



## 5.4 Limit coordinates

The results so far have already provided us with quite some insight about what the sequences  $\vec{\Omega}(A)$  look like. By Lemma 5.1 we know that the set of values that occur in  $\vec{\Omega}(A)$  is finite. Moreover, by Theorem 5.5 we know exactly the values at successor coordinates. In particular, we know that if the value of  $\vec{\Omega}(A)$  at  $\xi$  is the same as at the successor coordinate, then it remains the same for all further successors.

The question remains what happens at limit ordinals coordinates. In this subsection we shall determine at what limit ordinals a new value can be attained and how the new value relates to previous values. Let us start out the analysis by formulating a negative version of Theorem 5.7.

**Lemma 5.8.** *For  $\zeta < \xi$  we have that*

$$\begin{aligned} \Omega_\zeta(A) > \Omega_\xi(A) &\Leftrightarrow \\ (\exists \eta \in h_\zeta(A) \ \zeta \leq \eta < \xi) \vee (\exists \eta \in h_\zeta(A) \ \text{CNP}(\zeta, \eta) < \text{CNP}(\xi, \eta)). \end{aligned}$$

*Proof.* By contraposing equivalence (1.)  $\Leftrightarrow$  (5.) of Theorem 5.7 we get

$$\Omega_\zeta(A) \neq \Omega_\xi(A) \Leftrightarrow h_\xi(A) \neq h_\zeta(A) \vee \exists \eta \in h_\zeta(A) \ \xi \not\sim_\eta \zeta.$$

But, as  $\zeta < \xi$  we see

$$h_\xi(A) \neq h_\zeta(A) \Leftrightarrow \exists \eta \in h_\zeta(A) \ \zeta \leq \eta < \xi.$$

Likewise,

$$\exists \eta \in h_\zeta(A) \ \xi \not\sim_\eta \zeta \Leftrightarrow \exists \eta \in h_\zeta(A) \ \text{CNP}(\zeta, \eta) \neq \text{CNP}(\xi, \eta).$$

As  $\zeta < \xi$  we have

$$\text{CNP}(\zeta, \eta) \neq \text{CNP}(\xi, \eta) \Leftrightarrow \text{CNP}(\zeta, \eta) < \text{CNP}(\xi, \eta).$$

□

The first question to ask is at which limit coordinates the sequence  $\vec{\Omega}(A)$  can change. Let us first write precisely what it means for the sequence  $\vec{\Omega}(A)$  to change at some coordinate  $\zeta$ . We express this by the expression

$$\begin{aligned} \text{Change}(\zeta, A) &:= \\ \exists \xi < \zeta (\Omega_\xi(A) > \Omega_\zeta(A) \ \& \ \forall \eta (\xi \leq \eta < \zeta \Rightarrow \Omega_\xi(A) = \Omega_\eta(A))). \end{aligned}$$

The next lemma gives an alternative characterization of  $\text{Change}(\zeta, A)$ .

**Lemma 5.9.**  $\text{Change}(\zeta, A) \Leftrightarrow \forall \xi < \zeta \ \Omega_\xi(A) > \Omega_\zeta(A)$

*Proof.* For  $\zeta \in \text{Succ}$  this is clear. If  $\zeta \in \text{Lim}$ , then  $\{\Omega_\xi(A) \mid \xi < \zeta\}$  is a finite set as all the  $\Omega_\xi(A) \in \text{On}$  and these are weakly decreasing. Thus, at some point below  $\zeta$  the sequence must stabilize. □

We can now characterize at what limit ordinals the sequence  $\vec{\Omega}(A)$  can change.

**Theorem 5.10.** *For  $\zeta \in \text{Lim}$ :  $\text{Change}(\zeta, A) \Leftrightarrow \exists \xi \in h_\zeta(A) \zeta \in \text{CNA}(\xi)$*

*Proof.* For  $\zeta \in \text{Lim}$  we see that, by Lemma 5.9,  $\text{Change}(\zeta, A)$  is equivalent to the claim that, given  $\xi < \zeta$ ,  $\Omega_\xi(A) > \Omega_\zeta(A)$ .

By Lemma 5.8, the latter is in turn equivalent to

$$\forall \xi < \zeta (\exists \eta \in h_\xi(A) \xi \leq \eta < \zeta \vee \exists \eta \in h_\xi(A) \text{CNP}(\xi, \eta) < \text{CNP}(\zeta, \eta)),$$

or equivalently,

$$\forall \xi (\xi_0 < \xi < \zeta \rightarrow \exists \eta \in h_\zeta(A) \text{CNP}(\xi, \eta) < \text{CNP}(\zeta, \eta)),$$

where  $\xi_0 := \max\{\xi' \in A \mid \xi' < \zeta\}$ . Note that for these  $\xi$ , indeed, we have  $h_\xi(A) = h_\zeta(A)$ . We now claim that the latter is equivalent to  $\exists \eta \in h_\zeta(A) \zeta \in \text{CNA}(\eta)$ . Clearly, if  $\zeta \in \text{CNA}(\eta)$  for some  $\eta \in h_\zeta(A)$ , then  $\xi \downarrow \eta < \zeta \downarrow \eta$  for each  $\xi < \zeta$ .

For the converse direction, suppose  $\zeta \notin \text{CNA}(\eta)$  for all  $\eta \in h_\zeta(A)$ . Then, for all  $\xi'$  with

$$\max \bigcup_{\eta \in h_\zeta(A)} \text{CNA}(\eta) < \xi' < \zeta$$

we have  $\xi' \sim_\eta \zeta$  for all  $\eta \in h_\zeta(A)$ , whence by Theorem 5.7  $\Omega_{\xi'}(A) = \Omega_\zeta(A)$ .  $\square$

Now that we have fully determined at which limit coordinates a change can occur the only thing left to establish is the size of the change. In other words, if  $\text{Change}(\zeta, A)$  for some  $\zeta \in \text{Lim}$ , how does  $\Omega_\zeta(A)$  relate to  $\Omega_\xi(A)$  for  $\xi < \zeta$ ?

Here, our functions  $e^\xi$  come back into play:

**Theorem 5.11.** *Let  $\zeta \in \text{Lim}$ , and let  $\xi < \zeta$  be such that, whenever  $\xi' \in [\xi, \zeta)$ , it follows that  $\Omega_\xi(A) = \Omega_{\xi'}(A)$ . Then, for  $\theta \in [\xi, \zeta)$  large enough we have that*

$$\Omega_\theta(A) = e^{-\theta+\zeta} \Omega_\zeta(A) = e_{\ell\zeta} \Omega_\zeta(A).$$

*Proof.* As the values of  $\Omega_{\xi'}(A)$  do not change for  $\xi \leq \xi' < \zeta$  we know in particular by Theorem 5.7 that  $h_\xi(A) = h_{\xi'}(A)$  whence also

$$h_{\xi'}(A) = h_\zeta(A) \quad \text{for each } \xi' \in [\xi, \zeta]. \quad (2)$$

As  $\zeta = \zeta' + \omega^{\ell\zeta}$  we have that  $-\xi + \zeta \geq \omega^{\ell\zeta}$ . So certainly  $-\theta + \zeta = \omega^{\ell\zeta}$  for  $\theta \in [\xi, \zeta)$  large enough. Let  $\delta = -\theta + \zeta = \theta \downarrow \zeta = \omega^{\ell\zeta}$ .

Then,

$$\begin{aligned} \Omega_\theta(A) &= \text{Lemma 5.2} \\ o_\theta h_\theta(A) &= \text{By (2)} \\ o_\theta h_\zeta(A) &= \text{Lemma 3.13.4} \\ o(\theta \downarrow h_\zeta(A)) &= \text{Lemma 3.7.6} \\ o(\delta \uparrow (\zeta \downarrow h_\zeta(A))) &= \text{Lemma 3.13.3} \\ e^\delta o(\zeta \downarrow h_\zeta(A)) &= \text{Lemma 3.13.4} \\ e^\delta o_\zeta h_\zeta(A) &= \text{Lemma 5.2} \\ e_{\ell\zeta} \Omega_\zeta(A). & \end{aligned}$$

$\square$

Note that this theorem establishes the size of limit coordinates both in case a change does occur and in case no change occurs. The latter case can only be so when  $\Omega_\zeta(A)$  is a fixed point of  $e_{\ell\zeta}$ .

## 6 From local to global

The previous section has established exactly where changes occur in the  $\vec{\Omega}(A)$  sequences. Moreover, it established the size of each change in the sequence. We have distinguished two cases: successor coordinates and limit coordinates.

In Theorem 5.11 we have seen that the value of a limit coordinate fully determines its ‘direct predecessor’ and vice versa. Recall that the value of a successor coordinate is fully determined by the value of its predecessor but not vice versa. Thus, the values of the early coordinates fully determines what comes after it but not so in the other direction. In Section 3 we provided a calculus to compute  $o(A)$  for given  $A$ . Thus, the results in this section provides sufficient results to fully calculate  $\vec{\Omega}(A)$ .

However, the algorithm implicit in the current results are of a nature that all computations are performed globally: If we wish to compute  $\Omega_\zeta(A)$ , we need to compute the values of all its predecessors. Thus, first we compute  $\Omega_0(A) = o_0(A)$ , next we determine at what coordinates the sequence  $\vec{\Omega}(A)$  changes up to  $\zeta$ . In the end we compute all the successive values of the coordinates where  $\vec{\Omega}(A)$  changes to finally obtain  $\Omega_\zeta(A)$ .

We shall now see that each change in  $\vec{\Omega}(A)$  is of similar nature so that successively computing the changes corresponds to a certain transfinite iteration. Recall that  $\Omega_{\xi+1}(A) = \ell\Omega_\xi(A)$  by Theorem 5.5. We can see  $\ell$  as a natural left inverse of  $e^1 = e^{-\xi+(\xi+1)}$  so that

$$\begin{aligned} e^1\Omega_{\xi+1}(A) &= \Omega_\xi(A) \\ &\Rightarrow \\ \ell^1 e^1\Omega_{\xi+1}(A) &= \ell^1\Omega_\xi(A) \\ &\Rightarrow \\ \Omega_{\xi+1}(A) &= \ell\Omega_\xi(A). \end{aligned}$$

If, more generally, for every  $\vartheta$  we find an analogous left inverse  $\ell^\vartheta$  for  $e^\vartheta$ , then we may similarly obtain

$$\begin{aligned} e^{-\xi+\zeta}\Omega_\zeta(A) &= \Omega_\xi(A) \\ &\Rightarrow \\ \ell^{-\xi+\zeta}e^{-\xi+\zeta}\Omega_\zeta(A) &= \ell^{-\xi+\zeta}\Omega_\xi(A) \\ &\Rightarrow \\ \Omega_{\xi+1}(A) &= \ell^{-\xi+\zeta}\Omega_\xi(A) \end{aligned}$$

when  $\zeta, \xi$  and  $A$  are as in Theorem 5.11.

In [9] the authors systematically study natural left-inverses of hyperations and call them *cohyperations*. Once this is in place we can give a global calculus for our sequences, that is, a calculus that computes  $\Omega_\zeta(A)$  in ‘one step’ from  $\Omega_0(A)$  or from any other previous coordinate.

## 7 Hyperations and Cohyperations

In this section we shall briefly state the main definitions and results from [9] which are relevant for the current paper.

### 7.1 Hyperations

*Hyperation* is a form of transfinite iteration of normal functions. It is based on the additivity of finite iterations, that is  $f^{m+n} = f^m \circ f^n$  generalizing this to the transfinite setting.

**Definition 7.1** (Weak hyperation). *A weak hyperation of a normal function  $f$  is a family of normal functions  $\langle g^\xi \rangle_{\xi \in \text{On}}$  such that*

1.  $g^0 \xi = \xi$  for all  $\xi$ ,
2.  $g^1 = f$ ,
3.  $g^{\xi+\zeta} = g^\xi g^\zeta$ .

Par abuse de langage we will often write just  $g^\xi$  instead of  $\langle g^\xi \rangle_{\xi \in \text{On}}$ . Weak hyperations are not unique. However, if we impose a minimality condition, we can prove that there is a unique minimal hyperation.

**Definition 7.2** (Hyperation). *A weak hyperation  $g^\xi$  of  $f$  is minimal if it has the property that, whenever  $h^\xi$  is a weak hyperation of  $f$  and  $\xi, \zeta$  are ordinals, then  $g^\xi \zeta \leq h^\xi \zeta$ .*

*If  $f$  has a (unique) minimal weak hyperation, we call it the hyperation of  $f$  and denote it  $f^\xi$ .*

**Theorem 7.3.** *Every normal function  $f$  has a unique hyperation and it is given by the following recursion:*

1.  $f^0 \xi = \xi$ ,
2.  $f^1 = f$ ,
3.  $f^{\omega^\rho + \xi} = f^{\omega^\rho} f^\xi$ , where  $0 < \xi < \omega^\rho + \xi$ ,
4.  $f^{\omega^\rho} 0 = \lim_{\zeta \rightarrow \omega^\rho} f^\zeta 0$  for  $\rho > 0$ ,
5.  $f^{\omega^\rho} (\xi + 1) = \lim_{\zeta \rightarrow \omega^\rho} f^\zeta (f^{\omega^\rho} (\xi) + 1)$  for  $\rho > 0$ ,
6.  $f^{\omega^\rho} \xi = \lim_{\zeta \rightarrow \xi} f^{\omega^\rho} \zeta$  for  $\xi \in \text{Lim}$  and  $\rho > 0$ .

Moreover, there turns out to be a close connection between hyperations and Veblen progressions as shown by the following two theorems.

**Theorem 7.4.** *Let  $f$  be a normal function and let  $f_\alpha$  be the Veblen progression based on it. Given an ordinal  $\alpha$ , we have that  $f^{\omega^\alpha} = f_\alpha$ .*

**Theorem 7.5.** *Let  $g^\xi$  be a weak hyperation of a normal function  $f$ . If we moreover have that  $g^{\omega^\alpha} = f_\alpha$  for each  $\alpha$  then  $g^\xi = f^\xi$ .*

We will call the functions  $e^\alpha$  hyperexponentials. They can be used to define weak normal forms. For example, given an ordinal  $\xi$ , we say an expression

$$\xi = \sum_{i < I} e^{\alpha_i} \beta_i + n$$

is a *Weak Hyperexponential Normal Form* if  $I, n < \omega$ , and for each  $i + 1 < I$ , both  $e^{\alpha_i} \beta_i \geq e^{\alpha_{i+1}} \beta_{i+1}$  and  $\beta_i < e^{\alpha_i} \beta_i$ . Note that Weak Hyperexponential Normal Forms are typically not unique. For example  $\omega^\omega = e^2 1 = e^1 \omega$ . We do, however, have uniqueness if every  $\alpha_i$  is of the form  $\omega^\delta$ .

**Lemma 7.6.** *Every ordinal  $\xi > 0$  has a weak hyperexponential normal form.*

*If we further require that every exponent be of the form  $\omega^\delta$ , then the WHNF obtained is unique.*

*Proof.* Write  $\xi$  in Veblen Normal Form and replace  $\varphi_\alpha(\beta)$  by  $e^{\omega^\alpha}(1 + \beta)$  for  $\alpha > 0$ ,  $\varphi_0(\beta)$  by  $e^1(\beta)$  for  $\beta > 0$ . The occurrences of  $\varphi_0(0)$  can be captured in the term  $+n$  in the end of a WHNF.

If all exponents are of the form  $\omega^\delta$ , we may invert the process to obtain a VNF from a given WHNF; the uniqueness of the latter follows from the uniqueness of the former.  $\square$

## 7.2 Cohyperations

Hyperations are injective and hence invertible on the left; however, the inverse of a hyperation is typically not a hyperation, but a different form of transfinite iteration we call *cohyperation*. Instead of iterating normal functions we shall consider *initial functions*. We will say a function  $g$  is *initial* if, whenever  $I$  is an initial segment (i.e., of the form  $[0, \beta)$  for some  $\beta$ ), then  $f(I)$  is an initial segment. It is easy to see that  $f\xi \leq \xi$  for initial functions  $f$ .

**Definition 7.7** (Cohyperation). *A weak cohyperation of an initial function  $f$  is a family of initial functions  $\langle g^\xi \rangle_{\xi \in \text{On}}$  such that*

1.  $g^0 \xi = \xi$  for all  $\xi$ ,
2.  $g^1 = f$ ,
3.  $g^{\xi+\zeta} = g^\zeta g^\xi$ .

*If  $g$  is maximal in the sense that  $g^\xi \zeta \geq h^\xi \zeta$  for every weak cohyperation  $h$  of  $f$  and all ordinals  $\xi, \zeta$ , we say  $g$  is the cohyperation of  $f$  and write  $f^\xi = g^\xi$ .*

Both hyperations and cohyperations are denoted using the superscript; however, this does not lead to a clash in notation as the only function that is both normal and initial is the identity.

There is a general recursive scheme to compute actual cohyperations in the spirit of Theorem 7.3.

**Lemma 7.8.** *Every initial function  $f$  has a unique cohyperation, given by*

1.  $f^0\alpha = \alpha$ ,
2.  $f^1 = f$ ,
3.  $f^{\omega^\rho + \xi} = f^\xi f^{\omega^\rho}$  provided  $\xi < \omega^\rho + \xi$ ,
4.  $f^{\omega^\rho} \xi = f^{\omega^\rho} f^\eta \xi$ , if  $f^\eta \xi < \xi$  and  $\eta < \omega^\rho$ ,
5.  $f^{\omega^\rho} \xi = \sup_{\zeta < \xi} (f^{\omega^\rho} \zeta + 1)$ , if  $f^\eta \xi = \xi$  for all  $\eta < \omega^\rho$ , with  $\rho > 0$ .

At first glance it is not even clear that  $f^\xi$  is well defined in that it is single valued. In Item 4., there might be various  $\eta$ 's below  $\omega^\rho$  so that  $f^\eta \xi < \xi$ . In [9] it is shown that it does not matter which  $\eta$  one takes.

Let  $f$  be a normal function. Then,  $g$  is a *left adjoint* for  $f$  if, for all ordinals  $\alpha, \beta$ ,

1. if  $\alpha = f(\beta)$ , then  $g(\alpha) = \beta$  and
2. if  $\alpha < f(\beta)$ , then  $g(\alpha) < \beta$ .

Left-adjoints are natural left-inverses and cohyperating them yields left-adjoints to the corresponding hyperations in a uniform way:

**Theorem 7.9.** *Given a normal function  $f$  with left adjoint  $g$  and ordinals  $\xi < \zeta$  and  $\alpha$ ,  $g^\xi f^\zeta = f^{-\xi+\zeta}$  and  $g^\zeta f^\xi = g^{-\xi+\zeta}$ .*

**Theorem 7.10.** *The function  $\ell$  is a left adjoint to  $e$ , and thus  $\ell^\xi$  is left adjoint to  $e^\xi$  for all  $\xi$ .*

For the cohyperation of  $\ell$  we give the following easy recursive scheme.

**Theorem 7.11.** *For ordinals  $\xi, \zeta$ , the value of  $\ell^\xi \zeta$  is given by the following recursion:*

1.  $\ell^0\alpha = \alpha$ ,
2.  $\ell^\xi n = 0$  for  $n \in \omega$  and  $\xi > 0$ ,
3.  $\ell^\xi(\alpha + \omega^\beta) = \ell^\xi \omega^\beta$  if  $\xi > 0$ ,
4.  $\ell^{\omega^\rho + \xi} = \ell^\xi \ell^{\omega^\rho}$  provided  $\xi < \omega^\rho + \xi$ ,
5.  $\ell^{\omega^\rho} e^{\omega^\beta} \xi = \begin{cases} e^{\omega^\beta} \xi & \text{if } \omega^\rho < \omega^\beta, \\ \xi & \text{if } \omega^\rho = \omega^\beta, \\ \ell^{\omega^\rho} \xi & \text{in case } \omega^\rho > \omega^\beta. \end{cases}$

*Proof.* We shall first see that the recursive scheme of the unique cohyperation of  $\ell$  as given in Lemma 7.8 satisfies the recursion of the current theorem. Next, we shall see that the recursion of this theorem has a unique solution. The latter is necessary as we note that it is not fully determined how the last item of the recursion is to be applied, as an ordinal  $\zeta$  might be representable as  $e^{\omega^\beta}\xi$  in various ways using different  $\beta$  and  $\xi$ .

That  $\ell^0\alpha = \alpha$  follows directly from Lemma 7.8. Any  $\xi > 0$  can be written as  $1 + \xi'$  so that

$$\ell^\xi(\alpha + \omega^\beta) = \ell^{1+\xi'}(\alpha + \omega^\beta) = \ell^{\xi'}\ell(\alpha + \omega^\beta) = \ell^{\xi'}\ell\omega^\beta = \ell^{1+\xi'}\omega^\beta = \ell^\xi\omega^\beta.$$

From this, it directly follows that  $\ell^\xi n = 0$  for any  $\xi > 0$  and  $n \in \omega$ . Item 4 of the recursion holds trivially. Item 5 follows directly from Theorem 7.10 and Theorem 7.9.

We shall now show unicity. It is clear that we only need to focus on Item 5. Thus, we consider  $\ell^{\omega^\rho}e^{\omega^\beta}\xi$ . In [8] it is shown that there is a maximal  $\alpha$  such that  $e^{\omega^\beta}\xi = e^\alpha\zeta$  for some  $\zeta$ . We shall prove that  $\ell^{\omega^\rho}e^{\omega^\beta}\xi = \ell^{\omega^\rho}e^\alpha\zeta$ .

Let  $\omega^{\alpha_1} + \dots + \omega^{\alpha_n} =_{\text{CNF}} \alpha$  for this particular  $\alpha$ . By maximality of  $\alpha$ , we see that  $\beta \leq \alpha_1$ . In case  $\beta < \alpha_1$  we see by Theorem 7.4 that  $e^{\omega^\alpha}\zeta$  is a fixpoint of  $e^{\omega^\beta}$  so that

$$e^{\omega^\beta}\xi = e^\alpha\zeta = e^{\omega^\beta}e^{\omega^\alpha}\zeta = e^{\omega^\alpha}\zeta,$$

whence also  $\ell^{\omega^\rho}e^{\omega^\beta}\xi = \ell^{\omega^\rho}e^\alpha\zeta$ .

In case  $\beta = \alpha_1$ , we see also have that  $\ell^{\omega^\rho}e^{\omega^\beta}\xi = \ell^{\omega^\rho}e^\alpha\zeta$  as  $e^{\omega^{\alpha_1}}$  is injective.  $\square$

We will refer to the functions  $\ell^\xi$  as *hyperlogarithms*.

### 7.3 Exact sequences

A nice feature of cohyperations is that, in a sense, they need only be defined locally. To make this precise, we introduce the notion of an *exact sequence*.

**Definition 7.12.** Let  $g^\xi$  be a cohyperation, and  $f : \Lambda \rightarrow \Theta$  be an ordinal function.

Then, we say  $f$  is  $g$ -exact if, given ordinals  $\xi, \zeta$  with  $\xi + \zeta < \Lambda$ ,  $f(\xi + \zeta) = g^\zeta f(\xi)$ .

A  $g$ -exact function  $f$  describes the values of  $g^\xi f(0)$ . However, for  $f$  to be  $g$ -exact, we need only check a fairly weak condition:

**Lemma 7.13.** The following are equivalent:

1.  $f$  is  $g$ -exact
2. for every ordinal  $\xi$ ,  $f(\xi) = g^\xi f(0)$
3. for every ordinal  $\zeta > 0$  there is  $\xi < \zeta$  such that  $f(\zeta) = g^{-\xi+\zeta} f(\xi)$ .

## 8 A global characterization

In this section we shall unify the results obtained so far by describing the sequences  $\vec{\Omega}(A)$  using hyperexponentials and -logarithms.

**Theorem 8.1.** *Let  $A$  be a worm.*

*Then,  $\vec{\Omega}(A)$  is the unique  $\ell$ -exact sequence with  $\Omega_0(A) = o(A)$ .*

*Proof.* In view of Lemma 7.13, it suffices to show that, given any ordinal  $\zeta$ , there is  $\xi < \zeta$  such that  $\Omega_\zeta(A) = \ell^{-\xi+\zeta}\Omega_\xi(A)$ .

If  $\zeta$  is a successor ordinal, write  $\zeta = \xi + 1$ . Then, by Theorem 5.5, we have that  $\Omega_\zeta(A) = \ell\Omega_\xi(A)$ .

Meanwhile, if  $\zeta$  is a limit ordinal, we know from Lemma 5.11 that, for  $\xi < \zeta$  large enough,

$$\Omega_\xi(A) = e^{-\xi+\zeta}\Omega_\zeta(A).$$

Applying  $\ell^{-\xi+\zeta}$  on both sides and using Theorem 7.9, we see that

$$\ell^{-\xi+\zeta}\Omega_\xi(A) = \Omega_\zeta(A).$$

Thus we can use Lemma 7.13 to see that  $\vec{\Omega}(A)$  is  $\ell$ -exact, so that, for all  $\xi$ ,

$$\Omega_\xi(A) = \ell^\xi\Omega_0(A) = \ell^\xi o_0(A),$$

as claimed.  $\square$

Notice by Theorems 8.1 and 7.11 that the computations in omega sequences are rather easy if we have written the values in Weak Hyperexponential Normal Form (see Lemma 7.6) and are determined by the last term. If, for example,  $\Omega_\xi(A) = \alpha + e^{\omega^\zeta}(\beta)$ , then the next value where the  $\vec{\Omega}(A)$  sequence changes will be in  $\xi + \omega^\zeta$  jumping to the new value  $\Omega_{\xi+\omega^\zeta}(A) = \beta$ .

Further, hyperexponentials give us *lower bounds* on  $\ell$ -exact sequences. The value of  $\Omega_\xi(A)$  fully determines the values of  $\Omega_\zeta(A)$  for  $\zeta > \xi$  but not vice versa. However for  $\zeta > \xi$  we do have a lower-bound on  $\Omega_\xi(A)$ :

**Theorem 8.2.** *Given a worm  $A$  and ordinals  $\xi, \zeta$ ,  $\Omega_\xi(A) \geq e^\zeta\Omega_{\xi+\zeta}(A)$ .*

*Proof.* Towards a contradiction, assume that there is a worm  $A$  and ordinals  $\xi < \zeta$  such that  $\Omega_\xi(A) < e^{-\xi+\zeta}\Omega_\zeta(A)$ . Then, by Theorem 7.10,  $\ell^{-\xi+\zeta}\Omega_\xi(A) < \Omega_\zeta(A)$ .

But this is impossible by Theorem 8.1, given that  $\ell^{-\xi+\zeta}\Omega_\xi(A) = \Omega_\zeta(A)$ .  $\square$

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